Indian Statistical Institute, Bangalore

B. Math.(Hons.) II Year, First Semester

Semestral Examination

Analysis -III

Time: 3 hours | December 3rd, 2009 | Instructor: B.Rajeev

Maximum Marks 60

1. Prove Green's theorem for a domain $D \subset \mathbb{R}^2$ and vector fields (P, Q); state your assumptions on D and (P, Q). [10]

2. Show that the value of the sphere in $\mathbb{R}^3$ of radius a and centre at the origin is $V(a) = \frac{4}{3}\pi\ a^3$. [10]

3. Let $G \subset \mathbb{R}^3$ be the set $S = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$.

   a) If G is a twice continuously differentiable function on S and if $\overrightarrow{F} =$ curl G than $\bigtriangledown \cdot \overrightarrow{F} = 0$. [3]

   a) If $\overrightarrow{F} = (F_1, F_2, F_3)$ is a continuously differentiable function on S such that $\bigtriangledown \cdot \overrightarrow{F} = 0$ then $\exists$ a differentiable function G on S such that $\overrightarrow{F} =$ curl G. [7]

4. Let $V(t) = \{(x, y, z) : x^2 + y^2 + z^2 \leq t^2\}$ and $S(t) = \{(x, y, z) : x^2 + y^2 + z^2 = t^2\}, t > 0$. Let $F = (F_1, F_2, F_3)$ be a continuously differentiable vector field on $V(1)$. Then show that

   $\bigtriangledown \cdot \overrightarrow{F}(0) = \lim\limits_{t \downarrow 0} \frac{1}{|V(t)|} \int\int\limits_{S(t)} \overrightarrow{F} \cdot \overrightarrow{n}$ ds, where $\mid V(t) \mid=$ volume of $V(t)$ and $\overrightarrow{n}$ is the unit outward normal to $S(t)$. [10]

5. Let $F(x, y, z) = (y^2 \cos(xz), x^3 e^{yz}, -e^{-xyz})$. Evaluate $\int\int\limits_{S}$ curl $F \cdot n$ dS, where S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying above the xy plane and $\overrightarrow{n}$ is a suitable choice of the unit normal on S. [15]

6. Let $\overrightarrow{r} : T \longrightarrow S$ be a parametrization of a surface S and $T \subset \mathbb{R}^2$ be a bounded, open and connected set. Let $E \equiv E(u, v) :=\parallel \frac{\partial \overrightarrow{r}}{\partial u} \parallel^2; F \equiv F(u, v) := \frac{\partial \overrightarrow{r}}{\partial u} \cdot \frac{\partial \overrightarrow{r}}{\partial v}$ and $G \equiv G(u, v) :=\parallel \frac{\partial \overrightarrow{r}}{\partial v} \parallel^2$.

   a) Show that

   Area of $S = \int\int\limits_{T} \sqrt{EG - F^2} du\ dv$. [5]

   b) Let $T = (0, 2\pi) \times (0, 2\pi), b > t > 0$. Let $\overrightarrow{r} : T \longrightarrow S$ be $\overrightarrow{r}(\theta, \psi) = ((b + t\cos\theta)\cos\psi, (b + t\cos\theta)\sin\psi, t\sin\theta)$. Show that the area of S is $4\pi^2 tb$. [10]